ON THE EXISTENCE AND UNIQUENESS OF WEAK SOLUTIONS FOR A VORTICITY SEEDING MODEL

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Abstract. In this paper we study the Navier-Stokes equations with a Navier-type boundary condition that has been proposed as an alternative to common near wall models. The boundary condition we study, involving a linear relation between the tangential part of the velocity and the tangential part of the Cauchy stress-vector, is related to the vorticity seeding model introduced in the computational approach to turbulent flows. The presence of a point-wise non vanishing normal flux may be considered as a tool to avoid the use of phenomenological near wall models, in the boundary layer region. Furthermore, the analysis of the problem is suggested by recent advances in the study of Large Eddy Simulation.

In the two dimensional case we prove existence and uniqueness of weak solutions, by using rather elementary tools, hopefully understandable also by applied people working on turbulent flows. The asymptotic behaviour of the solution, with respect to the averaging radius δ, is also studied. In particular, we prove convergence of the solutions toward the corresponding solutions of the Navier-Stokes equations with the usual no-slip boundary conditions, as the small parameter δ goes to zero.

Key words. Navier-Stokes equations, boundary models for turbulent flows, existence, uniqueness, LES models.

AMS subject classifications. 35Q30, 76F75, 76B03.

1. Introduction. In this paper we consider the Navier-Stokes equations and in particular the role of boundary conditions in the simulation of boundary effects in turbulent flows. We consider the Navier–Stokes equations (in non-dimensional form) for viscous incompressible fluids in a bounded smooth domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$,

$$
\begin{cases}
\partial_t u - \dfrac{2}{Re} \nabla \cdot \mathbb{D}(u) + (u \cdot \nabla)u + \nabla p = 0 & \text{in } \Omega \times (0, T) \\[2mm]
\nabla \cdot u = 0 & \text{in } \Omega \times (0, T).
\end{cases}
\tag{1.1}
$$

We recall that $u = (u_1, \ldots, u_n)$ is the unknown velocity field, p is the hydrostatic pressure, $Re > 0$ is the Reynolds number, and $\mathbb{D}(u)$ is the deformation tensor, *i.e.*, the symmetric part of the matrix of derivatives of u:

$$
\mathbb{D}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right),
$$

and the Navier-Stokes equations are generally equipped with the *no-slip* boundary conditions on $\Gamma = \partial \Omega$

$$
u = 0 \quad \text{on } \Gamma \times (0, T).
\tag{1.2}
$$

To introduce the problem that we will study, we recall that while at a free surface it is natural to require continuity of the stress-tensor ($\mathbb{I}$ being the identity)

$$
\mathbb{T}(u, p) = -p\,\mathbb{I} + \frac{2}{Re}\mathbb{D}(u),
$$

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the conditions at a solid wall are much troublesome. The no-slip condition (1.2) has
been justified by Stokes [30] since the contrary assumption

> ... implies an infinitely greater resistance to the sliding of one portion
> of fluid past another than to the sliding of fluid over a solid.

It is well-known that there are situation in which the boundary condition (1.2) may
not be valid. For instance in Serrin [28] §64 it is pointed out that in high altitude
aerodynamics, or more generally when moderate pressure and low surface stresses
are involved, the adherence condition is no longer true, see also the review in Trues-
dell [31]. In this respect several authors proposed various *slip* (generally nonlinear)
conditions, modeling precise physical situations; see for instance Serrin [28], Beavers
and Joseph [3], Krein and Laptev [18].

From the historical point of view, the slip (with friction) boundary condition
proposed[1] by Navier [23] was

$$u \cdot \underline{n} = 0 \quad \text{and} \quad \beta \, u_\tau + \underline{\mathcal{T}}(u,p) = 0, \quad \beta > 0, \quad \text{on } \Gamma \times (0,T), \qquad (1.3)$$

where $\underline{n}$ denotes the unit normal vector to Γ, $u_\tau = u - (u \cdot \underline{n})\underline{n}$, while $\underline{\mathcal{T}}(u,p) = \underline{t}(u,p) - (\underline{t}(u,p) \cdot \underline{n})\,\underline{n}$ denotes the tangential part of the *Cauchy stress vector* $\underline{t}$ defined
by:

$$\underline{t}(u,p) = \underline{n} \cdot \mathbb{T}(u,p) = \sum_{k=1}^{n} \mathbb{T}_{ik}(u,p)\underline{n}_k.$$

The controversial between the condition (1.3) proposed by Navier, versus the Dirich-
let (1.2) proposed by Stokes was analysed also by Maxwell, who observed that the
same conditions may be derived in the kinetic theory of gases and that the parameter
β should depend on the Reynolds number Re and on the mean free-path λ, satisfying
the couple of consistency conditions

$$\beta \to \infty \text{ as } \lambda \to 0, \text{for } Re \text{ fixed}$$
$$\beta \to 0 \text{ as } Re \to \infty, \text{for } \lambda \text{ fixed}.$$

With the above asymptotics it possible to recover in both cases the correct no-slip
boundary condition for viscous fluids and the no-penetration condition for ideal fluids.
A study of the numerical problems related with the implementation of (1.3) can be
found in John [16].

Recently, Fujita [11] performed the analysis with the "slip or leak with friction"
boundary conditions. These conditions are of particular interest in the study of poly-
mers, blood flow, and flow through filters. The boundary conditions studied in [11]
with the techniques of variational inequalities, turn out to be particular cases of the
nonlinear boundary condition proposed at p. 240 in [28] and they are very strictly re-
lated to both the Navier and the no-slip boundary conditions. See also Consiglieri [8]
for similar problems.

Among other nonstandard boundary conditions we recall that studied by Begue *et
al.* [4] and the "do-nothing" Neumann conditions, very appealing for numerical studies,
implemented in Heywood *et al.* [15].

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[1]Note that in contrast to Stokes (1845) that used the continuum mechanics, Navier (1823) derived
the equations by using some formal analogy with the elasticity theory and the assumption that
molecules are animated by attractive and repulsive forces.

1.1. Near Wall Models and turbulent flows. Our interest in non standard boundary conditions comes essentially from the study of turbulent flows. First, we recall that in the boundary layer theory several log-law and power-law are introduced, together with the fictitious boundaries, in order to model turbulent flows within a small region near the boundary. Roughly speaking, appropriate nonlinear boundary conditions are imposed on an artificial boundary that lies inside the computational domain. The boundary conditions may simulate (at least in a computational approach) the behaviour of the boundary layer and they are modeled to take into account the peculiar behaviour of a fluid near the boundaries[2]

Our main motivation comes from the mathematical theory of Large Eddy Simulation (briefly, LES). In fact, the purpose of LES is to model the evolution of large coherent structures (eddies); this is done by studying the equations satisfied by a filtered velocity. Generally, the filtered velocity $\overline{u}$ is defined through a convolution

$$\overline{u}(x,t) = g_\delta(x) * u(x,t)$$

with a rapidly decreasing smoothing-kernel g_δ of width δ; in several cases of practical interest

$$g_\delta(x) = \left(\frac{\gamma}{\pi}\right)^{3/2} \frac{1}{\delta^3} \; \mathrm{e}^{-\frac{\gamma|x|^2}{\delta^2}}$$

By definition, the value of $\overline{u}$ at a point x_0 on the boundary Γ will depend on the behaviour of u in a neighborhood of width δ near that point; even if u is extended to zero for each $x \notin \overline{\Omega}$, it is clear that in general $\overline{u}(x_0) \neq 0$.

As pointed out in Galdi and Layton [14] the physical intuition may suggest that large coherent structures touching a wall do not penetrate, sliding along it, and losing their energy. The boundary condition of Navier may be revisited, by linking the micro-scale λ of the kinetic theory of gases, with the radius δ of the averaging filter.

Many Near Wall Models (NWM) have been tested in the computational approach, see Sagaut [27] and Piomelli and Balaras [26]. The results are not uniformly successful and a positive application is very often based on a fine tuning of parameters. This is why new models require at least a positive background from the physical hypotheses and a coherent mathematical analysis. In particular, a successful application of the Navier slip-with-friction boundary condition (1.3) is prevented by two main facts: 1) the presence of recirculation regions and 2) the presence of fast time-fluctuating quantities.

The first problem is motivated by the fact that in recirculation regions the local Reynolds number is very different from the main stream and it is natural to expect that β should depend (possibly in a nonlinear way) on a local Reynolds number related to the local slip speed, *i.e.*,

$$\beta = \beta(\delta, |u_\tau|).$$

Preliminary analysis has been performed by John *et al.* [17], Dunca *et al.* [10], and an appropriate power-law choice of β seems promising to improve the estimation of reattachment points.

The limitation of Navier law (1.3) in a Boundary Layer theory is that it can well describe time-averaged flow profiles, while the information coming from fluctuating

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[2]In this respect we quote J.C. Maxwell: "It is almost certain the the stratum of gas next to a solid body is in a very different state from the rest of the gas".

quantities in the wall-normal direction can play an important role in in triggering separation and detachment. To try to overcome this limitation, very recently Layton [19] recognized a particular class of boundary conditions, leading to conditions similar "in spirit" to the so called *vorticity seeding methods*. In fact, a Navier slip-with-friction boundary condition implies the generation of vorticity at the boundary, proportional to the tangential velocity. More precisely, in the case of a two-dimensional domain Ω, for each smooth function v such that $v \cdot \underline{\mathrm{n}} = 0$ on the boundary, it holds

$$\underline{\mathrm{n}} \cdot \mathbb{D}(v) \cdot \underline{\tau} - \frac{1}{2}\mathrm{curl}\, v + k(v \cdot \tau) = 0 \qquad \text{on } \Gamma,$$

where $\mathrm{curl}\, v = \partial v_1 / \partial x_2 - \partial v_2 / \partial x_1$, $\underline{\tau}$ denotes the unit tangent vector, while k is the curvature of Γ.

In particular, in [19] the following boundary condition is proposed to simulate the boundary effects

$$u \cdot \underline{\mathrm{n}} = \delta^2 g(x,t) \quad \text{and} \quad \frac{L}{\delta Re}u_\tau + \underline{\mathcal{I}}(u,p) = 0, \tag{1.4}$$

where g is a *highly oscillating* function in the time variable (hopefully a random variable in numerical tests), while it may be very smooth in the space variables and it should satisfy the natural compatibility condition

$$\int_\Gamma g(x,t)\,d\sigma = 0 \qquad \forall\, t \in (0,T), \tag{1.5}$$

which is required by the normal trace of a divergence-free vector field.

This way of reasoning is also similar to the introduction of stochastic fluctuations, to simulate the micro-scale effects. A comprehensive introduction to stochastic partial differential equations in fluid mechanics and the statistical approach can be found in Monin and Yaglom [21] and one main mathematical paradigm is that an additional non-smooth term on the right-hand side may naturally take into account the effect of the fast fluctuating quantities. This leads to study the system

$$\partial_t u - \frac{2}{Re}\nabla \cdot \mathbb{D}(u) + (u \cdot \nabla)u + \nabla p = f + \frac{\partial g}{\partial t},$$

where g is a function that does not have a proper time derivative, but is just continuous or with other weak properties. Appropriate notion of solutions, together with the statistical properties of the above problem are studied in Bensoussan and Temam [6] and Višik and Fursikov [32].

1.2. Setting of the problem. In the sequel we will restrict to the two dimensional case, since the nonlinear character of the equations imposes some restriction, see Remark 2.5. Furthermore, we fix the values of L and of the Reynolds number to 2 due to the fact that we will not deal with the singular limit $Re \to \infty$ (regarding this limit see also the recent works of Clopeau *et al.* [7] and Mucha [22]). In our case we will study the following boundary-initial value problem:

$$\begin{cases} \partial_t u - \nabla \cdot \mathbb{D}(u) + (u \cdot \nabla)u + \nabla p = f & \text{in } \Omega \times (0,T) \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0,T) \\ u \cdot \underline{\mathrm{n}} = \delta^\alpha g(x,t) & \text{on } \Gamma \times (0,T) \\ u \cdot \underline{\tau} + \delta\, \underline{\mathrm{n}} \cdot \mathbb{D}(u) \cdot \underline{\tau} = 0 & \text{on } \Gamma \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{1.6}$$

Note that a similar problem, involving the Smagorinsky-Ladyžhenskaya turbulence model together with a nonlinear dependence of β on u, has been studied for instance by Parés [25], but in this reference the normal datum g is not allowed to depend on the time variable. In the sequel our main interest will be to find weak hypotheses on $g(x,t)$ with respect to the time variable (without any essential restriction on the space regularity) that allow to prove existence of weak solutions to the Navier-Stokes equations, see Theorem 1.1. In particular, in our analysis we will focus on two main points:

    a) to show the existence of weak solutions in the sense of Leray-Hopf (since we do not want to deal with any weaker concept of solution);

    b) to use only elementary tools of functional analysis, in order to keep the paper intelligible to non specialists.

In other words, we want to consider solutions in an usual sense and we want also to interact with applied people interested in this problem, still keeping all the mathematical rigor needed to deal with the problem and a certain sharpness of the results. In the case of non-homogeneous no-slip conditions, several results of existence and uniqueness of suitable solutions can be found in Amann [2].

Our intent to have a non-vanishing normal datum can be heuristically understood also with the following argument: suppose that (for simplicity in two dimensions) we have a fictitious boundary Γ_1 and we want to impose a condition on it in order to resolve numerically the equation in a smaller domain $\Omega_1 \subset \Omega$, that rules out the boundary layer (see the figure below).
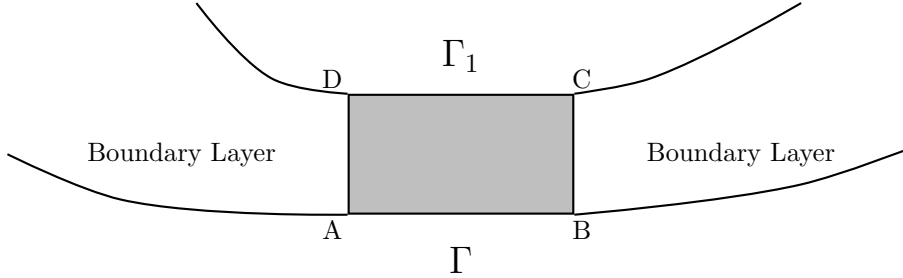

FIG. 1. *The fictitious boundary*

We have to require, by the incompressibility of the flow, that

$$\int_{\{ABCD\}} \nabla \cdot u \, dx = \int_{\partial\{ABCD\}} u \cdot \underline{n} \, d\sigma = 0,$$

for each (also curvilinear or infinitesimal) "rectangle" $\{ABCD\}$ touching the boundary Γ as in the figure. Since the behaviour of the flow is not known, in general we have[3]

$$\int_{\{CD\}} u \cdot \underline{n} \, ds = - \left[\int_{\{BC\}} u \cdot \underline{n} \, ds + \int_{\{DA\}} u \cdot \underline{n} \, ds \right] \neq 0.$$

This may justify the introduction of a non vanishing normal flux, also with very low regularity properties, namely the same shared by the trace of a turbulent flow in the boundary layer region.

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[3]Note that the line integral over the segment $\{AB\}$ vanishes, since on the "true boundary" Γ both the Navier or no-slip conditions impose that $u \cdot \underline{n} = 0$.

1.3. Main results. In this section we briefly enunciate the results we will prove, together with their precise and rigorous statement.

In the sequel Ω will a bounded, connected, open set in $\mathbb{R}^3$, locally situated on one side of its boundary Γ, a manifold of (at least) class $C^{1,1}$ (Lipschitz-continuous first derivatives). The existence of the unit outward normal $\underline{n}$ derives by results proved in Nečas [24].

The first result we will prove is an existence and uniqueness theorem for weak solutions of the Navier-Stokes system (1.6), with boundary conditions (1.4).

THEOREM 1.1. *Let be given* $g \in H^{\frac{1}{2}+\varepsilon}(0,T;H^{\frac{1}{2}}(\Gamma))$, *satisfying the compatibility condition* (1.5), $f \in L^2((0,T) \times \Omega)$, *and assume that* $u_0 \in L^2(\Omega)$, *with* $\nabla \cdot u_0 = 0$. *Then, there exists a unique weak solution*

$$u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),$$

of problem (1.6).

Next, we want to study the behaviour of the solution to problem (1.6) as the small parameter δ converges to zero. In view of the considerations of the previous section, one can expect that, as the boundary layer becomes thinner and thinner, the solutions will look more and more like the classical solutions corresponding to the *no-slip* boundary condition. Indeed, this is the case, as it is shown by the following theorem.

Let u_δ be the solution of (1.6) (we emphasize the dependence on δ in this framework) and let v be the solution to the Navier-Stokes equations with the same initial value and no-slip boundary conditions (to be more precise, the vector field v is the solution to system (3.2)). As we shall see in Section 3,

$$u_\delta = v + \mathcal{O}(\delta^{\frac{1}{3}}),$$

so that the "no-slip solution" represents the average behaviour, once one neglects the effect at the boundary. The term $u_\delta - v$ can be seen as the "fluctuation term", which takes into account the non-trivial dynamics at the boundary.

THEOREM 1.2. *Assume* $u_0 \in H^1(\Omega)$, *with* $\nabla \cdot u_0 = 0$, $g \in H^{\frac{1}{2}+\varepsilon}(0,T;H^{\frac{1}{2}}(\Gamma))$ *satisfying the compatibility condition* (1.5), *and* $f \in L^2((0,T) \times \Omega)$. *Then*

$$\sup_{0 \leq t \leq T} \|u_\delta - v\|^2 + \int_0^T \left(\|\mathbb{D}(u_\delta - v)\|^2 + \frac{1}{\delta}\|(u_\delta - v) \cdot \underline{\tau}\|_\Gamma^2 \right) dt = \mathcal{O}(\delta^{\frac{2}{3}}).$$

In particular, u_δ *converges to* v *in* $L^\infty(0,T;L^2(\Omega))$ *and* $L^2(0,T;H^1(\Omega))$.

2. A result of existence and uniqueness of weak solutions. In this section we prove Theorem 1.1. For the sake of simplicity, we consider the normal datum as

$$u \cdot \underline{n} = g(x,t),$$

namely, dropping the dependence on δ, since it is not relevant in view of the existence and uniqueness result we are going to show. In the last section we shall see how to deal with a right-hand side that scales by a power of δ.

Let us consider the evolution problem

$$\begin{cases} \partial_t u - \nabla \cdot \mathbb{D}(u) + (u \cdot \nabla)\, u + \nabla p = f & \text{in } \Omega \times (0,T) \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0,T) \\ u \cdot \underline{n} = g(x,t) & \text{on } \Gamma \times (0,T) \\ \delta\, \underline{n} \cdot \mathbb{D}(u) \cdot \underline{\tau} + u \cdot \underline{\tau} = 0 & \text{on } \Gamma \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \qquad (2.1)$$

with $f \in L^2((0,T) \times \Omega)$ and $\nabla \cdot f = 0$, just to avoid technicalities, and with g not very smooth, say

$$g \in H^{1/2+\epsilon}(0,T; H^{1/2}(\Gamma)) \tag{2.2}$$

satisfying the compatibility condition (1.5).

2.1. Function spaces. We use the classical Lebesgue spaces and in particular, we will work exclusively in the Hilbert framework, so we will use the space L^2. For simplicity we do not distinguish between space of scalar, vector, or either tensor valued functions, and the symbol $\|\,.\,\|$ will denote the norm in $L^2(\Omega)$. The norm in $L^2(\Gamma)$ will be denoted by $\|\,.\,\|_\Gamma$.

In the sequel we will use the customary Sobolev spaces for which we refer to Adams [1] and we will use the notion of "trace" over the boundary Γ of Ω. Mainly we will use the space $H^1(\Omega)$ with norm denoted by $\|\,.\,\|_{H^1}$ and its trace space $H^{1/2}(\Gamma)$, with norm $\|\,.\,\|_{1/2,\Gamma}$.

In addition, we define the spaces

$$H = \{\, u \in L^2(\Omega) \,|\, \nabla \cdot u = 0, \ u \cdot \underline{n} = 0 \text{ on } \Gamma \,\},$$

and

$$V = \{\, u \in H^1(\Omega) \,|\, \nabla \cdot u = 0, \ u \cdot \underline{n} = 0 \text{ on } \Gamma \,\},$$

and we endow V with the norm $\|u\|_V = \|\nabla u\|$. Moreover, we define the space of tangential vector fields as

$$H_\tau^1 = \{\, u \in H^1(\Omega) \,|\, u \cdot \underline{n} = 0 \text{ on } \Gamma \,\}.$$

2.1.1. Fractional derivative. In order to define properly the spaces we will use, we need also to define fractional derivatives. The fractional derivative may be defined through singular integrals

$$D_t^\alpha U(x,t) = \frac{d}{dt} \int_0^t \frac{U(s,x)}{(t-s)^\alpha}\, ds \quad \text{for } 0 \le \alpha < 1,$$

but for our purposes it is better to deal with a definition *via* the Fourier transform.

Given $\phi(x,t)$, defined for $t \in [0,T]$, with values in the Hilbert space $\mathbb{X}$ and integrable (in the Bochner sense), we define

$$\widetilde{\phi}(t,x) = \begin{cases} \phi(t,x) & \text{for } t \in [0,T] \\ 0 & \text{elsewhere,} \end{cases} \tag{2.3}$$

and its Fourier transform (with respect to the time variable) is

$$\widehat{\phi}(x,\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widetilde{\phi}(x,t)\, \mathrm{e}^{-\mathrm{i}t\xi}\, dt,$$

so that we can define the fractional Sobolev spaces of functions having $\alpha-$order derivative in L^2 :

$$H^\alpha(\mathbb{R}; \mathbb{X}) := \Big\{ f \in L^2(\mathbb{R}; \mathbb{X}) : \int_{\mathbb{R}} |\xi|^{2\alpha} \|\widehat{f}(\xi)\|_{\mathbb{X}}^2\, d\xi < \infty \Big\}.$$

2.2. The linear stationary problem. The first step to solve (2.1) is to consider the linear stationary problem

$$
\begin{cases}
-\nabla \cdot \mathbb{D}(G) + \nabla \Pi = 0 & \text{in } \Omega \times (0,T) \\
\nabla \cdot G = 0 & \text{in } \Omega \times (0,T) \\
G \cdot \underline{n} = g(x,t) & \text{on } \Gamma \times (0,T) \\
\delta\, \underline{n} \cdot \mathbb{D}(G) \cdot \underline{\tau} + G \cdot \underline{\tau} = 0 & \text{on } \Gamma \times (0,T),
\end{cases}
\tag{2.4}
$$

where the time variable is now just a parameter.

THEOREM 2.1. *Let be given $g \in H^{1/2+\epsilon}(0,T;H^{1/2}(\Gamma))$, satisfying the compatibility condition (1.5). Then, there exists a unique G solution of (2.4) such that*

$$
G(x,t) \in H^{1/2+\epsilon}(0,T;H^1(\Omega)).
\tag{2.5}
$$

Moreover, there is a constant C_0, depending only on Ω, such that

$$
\|\nabla G\| + \|\Pi\| \le C_0(1 + \delta^{-\frac{1}{2}})\|g\|_{H^{\frac{1}{2}}(\Gamma)}.
\tag{2.6}
$$

Proof. See Solonnikov and Ščadilov [29] and Beirão da Veiga [5]. In fact, for each t it is possible to solve a linear stationary Stokes problem (with the appropriate boundary conditions) that has a unique solution belonging to $H^1(\Omega)$. The regularity in the time variable is inherited by the function G.

We give a formal (but completely justified) argument for the estimate (2.6), following the approach to the existence in L^2-spaces introduced in Beirão da Veiga [5] to find appropriate estimates on G. We consider the bilinear form

$$
B(u,\phi) = \int_\Omega \mathbb{D}(u)\mathbb{D}(\phi)\,dx
$$

and the functions (G,Π), that solve (2.4), must satisfy

$$
B(G,\phi) - \int_\Omega \Pi \nabla \cdot \phi\,dx + \frac{1}{\delta}\int_\Gamma G \cdot \phi\,d\sigma = 0, \qquad \forall\, \phi \in H^1_\tau.
$$

In order to deal with the inhomogeneous problem, we introduce a vector field G_1 such that

$$
\begin{cases}
\nabla \cdot G_1 = 0 & \text{in } \Omega, \\
G_1 \cdot \underline{n} = g & \text{on } \Gamma, \\
\|G_1\|_{H^1} \le C\|g\|_{H^{\frac{1}{2}}(\Gamma)}.
\end{cases}
$$

The construction of such a vector field is rather standard and can be found for instance in Galdi [13].

By defining $G = G_1 + G_2$, the function G_2 must satisfy

$$
B(G_2,\phi) - \int_\Omega \Pi \nabla \cdot \phi\,dx + \frac{1}{\delta}\int_\Gamma G_2 \cdot \phi\,d\sigma = -B(G_1,\phi) - \frac{1}{\delta}\int_\Gamma G_1 \cdot \phi\,d\sigma,
$$

for each ϕ tangential to the boundary. If $\phi = G_2$, we get (since $G_1 \cdot \underline{n} = 0$ and $\nabla \cdot G_2 = 0$)

$$
\|\nabla G_2\|^2 + \frac{1}{\delta}\|G_2\|^2_\Gamma \le \|\nabla G_2\|\,\|\nabla G_1\| + \frac{1}{\delta}\|G_1\|_\Gamma\|G_2\|_\Gamma,
$$

and consequently

$$\frac{1}{2}\|\nabla G_2\|^2 + \frac{1}{2\delta}\|G_2\|_\Gamma^2 \le \frac{1}{2}\|\nabla G_1\|^2 + \frac{1}{2\delta}\|G_1\|_\Gamma^2.$$

This finally implies that

$$\|\nabla G_2\|^2 \le C\left(1 + \frac{1}{\delta}\right)\|g\|_{H^{\frac{1}{2}}(\Gamma)}^2,$$

where the constant C depends on Ω, but it is independent of δ. The estimate on the pressure can be obtained by approximation, by studying a slightly different equation (see [5] for details). $\square$

Indeed, notice that, for our aims, it is enough to have

$$G(x,t) \in H^{1/2+\epsilon}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)). \tag{2.7}$$

but currently we do not know the minimal assumption on g in order to have the above regularity. Regarding the usual no-slip boundary conditions see the result proved by Fursikov, Gunzburger, and Hou [12].

2.3. The linear evolution problem. The next step for the analysis of the nonlinear evolution problem (2.1) is the following linear evolution problem

$$\begin{cases} \partial_t z - \nabla \cdot \mathbb{D}(z) + \nabla q = 0 & \text{in } \Omega \times (0,T) \\ \nabla \cdot z = 0 & \text{in } \Omega \times (0,T) \\ z \cdot \underline{n} = g & \text{on } \Gamma \times (0,T) \\ \delta\, \underline{n} \cdot \mathbb{D}(z) \cdot \underline{\tau} + z \cdot \underline{\tau} = 0 & \text{on } \Gamma \times (0,T) \\ z(x,0) = G(x,0) & \text{in } \Omega. \end{cases} \tag{2.8}$$

We shall treat the non-linear problem as a perturbation of such a linear system. Let us introduce the new unknowns

$$Z(x,t) = z(x,t) - G(x,t) \quad \text{and} \quad Q(x,t) = q(x,t) - \Pi(x,t)$$

so that we are reduced to a homogeneous problem for the new unknowns (Z,Q):

$$\begin{cases} \partial_t Z - \nabla \cdot \mathbb{D}(Z) + \nabla Q = -\partial_t G & \text{in } \Omega \times (0,T) \\ \nabla \cdot Z = 0 & \text{in } \Omega \times (0,T) \\ Z \cdot \underline{n} = 0 & \text{on } \Gamma \times (0,T) \\ \delta\, \underline{n} \cdot \mathbb{D}(Z) \cdot \underline{\tau} + Z \cdot \underline{\tau} = 0 & \text{on } \Gamma \times (0,T) \\ Z(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{2.9}$$

The above problem is not completely standard, since the right-hand side does not satisfy the usual properties. For instance, one can note that $\partial_t G$ does not belong to the domain of the Stokes operator, since $\partial_t G \cdot \underline{n} \neq 0$. This is the main difficulty: the low regularity of this term can be treated in a more standard way, while the above fact is responsible for a different approach.

THEOREM 2.2. *Assume that (G,Π) is a solution to system* (2.4), *with G satisfying the regularity property* (2.7). *Then, there exists a unique solution (z,q) (where q is*

unique up to an additive constant not depending on the space variable) to system (2.8) *such that*

$$z \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \cap H^{\frac{1}{2}-\varepsilon}(0,T;L^2(\Omega)). \qquad (2.10)$$

Moreover,

$$\sup_{0\le t\le T} \|z(t)\|^2 + \int_0^T \left(\|\mathbb{D}(z)(t)\|^2 + \frac{1}{\delta}\|z(t)\cdot\underline{\tau}\|_\Gamma^2 \right) dt + \|z\|^2_{H^{\frac{1}{2}-\varepsilon}(0,T;L^2(\Omega))} \le$$

$$\le C \left(\|G\|^2_{H^{\frac{1}{2}+\varepsilon}(0,T;L^2(\Omega))} + \int_0^T \|\mathbb{D}(G)\|^2 \right). \quad (2.11)$$

Proof. By virtue of Theorem 2.1, it is enough to prove the same claim of this theorem on the solution (Z,Q) of problem (2.9). Since we just know that $\partial_t G \in H^{-\frac{1}{2}+\varepsilon}(0,T;L^2(\Omega))$, we introduce a sequence $G^N \in H^1(\mathbb{R};L^2(\Omega))$ of approximate functions such that
(a) $G^N|_{[0,T]} \longrightarrow G$ in $H^{\frac{1}{2}+\varepsilon}(0,T;L^2(\Omega))$, as $\to \infty$,
(b) $\|\partial_t G^N\|_{L^2(0,T;L^2(\Omega))} = N$.
The way to do this extension is rather standard: first we can define $\overline{G} : \mathbb{R} \to L^2(\Omega)$ with an extension by reflection. Then, we consider a sequence ρ_N of mollifiers and the function G_N will be the restriction on $[0,T]$ of the function $\rho_N * \overline{G}$.

The proof is based on the Faedo-Galerkin procedure. By Clopeau *et al.* [7], (but also the recent abstract results in [5]) we know that there exists a basis $(\phi_n)_{n\in\mathbb{N}}$ of functions in $H^3(\Omega)$ of the space V (and also of H), such that

$$\delta\,\underline{\mathrm{n}}\cdot\mathbb{D}(\phi_n)\cdot\underline{\tau} + \phi_n\cdot\underline{\tau} = 0.$$

Now, let $Z_n^N(t,x) = \sum_{k=1}^n \zeta_{n,k}^N(t)\phi_k(x)$ be the solution of the following (finite dimensional) linear system of Ordinary Differential Equations (ODE):

$$\begin{cases} \dfrac{d}{dt}\langle Z_n^N,\phi_k\rangle + \langle\mathbb{D}(Z_n^N),\mathbb{D}(\phi_k)\rangle + \dfrac{1}{\delta}\displaystyle\int_\Gamma (Z_n^N\cdot\underline{\tau})\,(\phi_k\cdot\underline{\tau})\,d\sigma = -\dfrac{d}{dt}\langle G^N,\phi_k\rangle \\ (Z_n^N(x,0),\phi_k) = 0, \end{cases}$$

for $t \in (0,T)$ and $k = 1,\dots,n$, where $\langle\cdot,\cdot\rangle$ denotes the usual L^2-scalar product. Notice that the divergence-free constraint and the boundary conditions on Z_n^N are automatically verified. By using a standard argument it is easy to prove that such a system of ODE has a unique solution $Z_n^N \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$; indeed, by multiplying each equation by the corresponding term $\zeta_{n,k}^N$, summing over k and integrating by parts over Ω, one easily obtains the following estimate

$$\sup_{0\le t\le T} \|Z_n^N(t)\|^2 + \int_0^T \left(\|\mathbb{D}(Z_n^N)\|^2 + \frac{1}{\delta}\int_0^T \|Z_n^N\|_\Gamma^2 \right) dt \le C\|G^N\|^2_{H^1(0,T;L^2(\Omega))},$$

for a constant C, depending only on Ω. Unfortunately, such an estimate, beside being uniform in n, is not uniform in N due to property *(b)* of the approximate sequence $(G^N)_{N\in\mathbb{N}}$. Hence, we need other *a-priori* estimates on the solutions Z_n^N of the finite dimensional problem. We continue working on the Z_n^N, since such functions are smooth enough for the computation that will be performed.

We multiply again the equations by the terms $\zeta_{n,k}^N$, sum over k and integrate by parts, but we estimate the right-hand side in the following way:

$$
\begin{aligned}
\sup_{0 \le t \le T} \|Z_n^N(t)\|^2 + \int_0^T \left(\|\mathbb{D}(Z_n^N)\|^2 + \frac{1}{\delta}\|Z_n^N\|_\Gamma^2 \right) dt &\le \Big| \int_0^T \int_\Omega \partial_t G^n \cdot Z_n^N \Big| \, dx dt \\
&\le \|\partial_t G^N\|_{H^{-\frac{1}{2}+\varepsilon}(0,T;L^2(\Omega))} \|Z_n^N\|_{H^{\frac{1}{2}-\varepsilon}(0,T;L^2(\Omega))} \\
&\le \|G^N\|_{H^{\frac{1}{2}+\varepsilon}(0,T;L^2(\Omega))} \|Z_n^N\|_{H^{\frac{1}{2}-\varepsilon}(0,T;L^2(\Omega))},
\end{aligned}
\tag{2.12}
$$

so that we only need to show a uniform estimate (with respect to both n and N) of Z_n^N in the space $H^{\frac{1}{2}-\varepsilon}(0,T;L^2(\Omega))$. We shall use the Fourier transform characterization of the norm of fractional Sobolev spaces (see Adams [1]) to get such estimate. Each $\widetilde{Z}_n^N$ (such functions have been defined in (2.3)) is a solution of the following equation

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega \widetilde{Z}_n^N \cdot \phi_k + \int_\Omega \mathbb{D}(\widetilde{Z}_n^N) \cdot \mathbb{D}(\phi_k) + \frac{1}{\delta}\int_\Gamma \widetilde{Z}_n^N \cdot \phi_k = \\
= -\frac{d}{dt}\int_\Omega \widetilde{G}^N \cdot \phi_k + \delta(t)\int_\Omega G^N(0) \cdot \phi_k - \delta(t-T)\int_\Omega (Z_n^N(T) + G^N(T)) \cdot \phi_k,
\end{aligned}
$$

for each $k = 1, \dots, n$, in the sense of distributions with respect to the time variable. Here $\delta(\cdot)$ is the usual Dirac's *delta function*. In the frequency Fourier variable ξ, the above equation reads:

$$
\begin{aligned}
-\mathrm{i}\xi \int_\Omega \widehat{Z}_n^N \cdot \phi_k + \int_\Omega \mathbb{D}(\widehat{Z}_n^N) \cdot \mathbb{D}(\phi_k) + \frac{1}{\delta}\int_\Gamma \widehat{Z}_n^N \cdot \phi_k = \\
= \mathrm{i}\xi \int_\Omega \widehat{G}^N \cdot \phi_k + \int_\Omega G^N(0) \cdot \phi_k - \mathrm{e}^{-\mathrm{i}\xi T}\int_\Omega (Z_n^N(T) + G^N(T)) \cdot \phi_k,
\end{aligned}
$$

see for instance Lions [20], where this tool is used to prove estimates on the fractional derivative of the solution. Note that in that reference, and in all involving fractional derivatives for the Navier-Stokes equations, the starting point is the existence of a weak solution and on it it is possible to prove additional estimates. In our case the existence of weak solution derives from the fractional derivative estimates and it seems not possible to prove the usual existence results.

Consequently, we get

$$
\begin{aligned}
-\mathrm{i}\xi\|\widehat{Z}_n^N(\xi)\|^2 + \|\mathbb{D}(\widehat{Z}_n^N)(\xi)\|^2 + \frac{1}{\delta}\|\widehat{Z}_n^N(\xi)\|_\Gamma^2 = \\
= \mathrm{i}\xi \int_\Omega \widehat{G}^N \cdot \overline{\widehat{Z}_n^N} + \int_\Omega G^N(0) \cdot \overline{\widehat{Z}_n^N} - \mathrm{e}^{-\mathrm{i}\xi T}\int_\Omega (Z_n^N(T) + G^N(T)) \cdot \overline{\widehat{Z}_n^N}.
\end{aligned}
$$

Take the imaginary part and multiply both sides of the previous formula by $|\xi|^{2\alpha-1}$, with $\alpha < \frac{1}{2}$ so that, by using Young's inequality, one gets

$$
|\xi|^{2\alpha}\|\widehat{Z}_n^N(\xi)\|^2 \le C|\xi|^{2\alpha}\|\widehat{G}^N\|^2 + C|\xi|^{2\alpha-2}(\|G^N(T)\| + \|Z_n^N(T)\| + \|G^N(0)\|)^2.
$$

In order to estimate the integral $\int_{\mathbb{R}} |\xi|^{2\alpha} \|\widehat{Z}_n^N(\xi)\|^2 \, d\xi$, we split it in two parts; by the above estimate,

$$\int_{|\xi|>1} |\xi|^{2\alpha} \|\widehat{Z}_n^N(\xi)\|^2 \leq$$

$$\leq C \int_{\mathbb{R}} |\xi|^{2\alpha} \|\widehat{G}^N\|^2 + C(\|G^N(T)\| + \|Z_n^N(T)\| + \|G^N(0)\|)^2 \int_{|\xi|>1} |\xi|^{2\alpha-2};$$

the first term on the right-hand side is controlled by $C\|G^N\|^2_{H^{\frac{1}{2}+\varepsilon}(0,T;L^2(\Omega))}$, while

$$\|Z_n^N(T)\|^2 \leq C\|G^N\|_{H^{\frac{1}{2}+\varepsilon}(0,T;L^2(\Omega))} \|Z_n^N\|_{H^{\frac{1}{2}-\varepsilon}(0,T;L^2(\Omega))},$$

by virtue of (2.12); $\|G^N(0)\|$ is bounded by $\|G(0)\|$, and finally, by using the Morrey inequality that implies $H^{1/2+\varepsilon}(0,T) \subset C([0,T])$ (see Adams [1]) we get

$$\|G^N(T)\| \leq \|G^N\|_{H^{\frac{1}{2}+\varepsilon}(0,T;L^2(\Omega))}.$$

The second part is estimated as follows, by using Parseval's theorem, Poincaré inequality and estimate (2.12),

$$\int_{|\xi|\leq 1} |\xi|^{2\alpha} \|\widehat{Z}_n^N\|^2 \, d\xi \leq \int_{\mathbb{R}} \|\widehat{Z}_n^N\|^2 \, d\xi = \int_0^T \|Z_n^N(t)\|^2 \, dt$$

$$\leq C \int_0^T \|\mathbb{D}(Z_n^N)\|^2 dt$$

$$\leq C\|G^N\|_{H^{\frac{1}{2}+\varepsilon}(0,T;L^2(\Omega))} \|Z_n^N\|_{H^{\frac{1}{2}-\varepsilon}(0,T;L^2(\Omega))}.$$

In conclusion, by collecting all of the above estimates, we finally get that, for each $\varepsilon \in (0, \frac{1}{2})$, there exists a constant C, depending only on Ω and ε, such that

$$\|Z_n^N\|_{H^{\frac{1}{2}-\varepsilon}(0,T;L^2(\Omega))} \leq C\|G^N\|_{H^{\frac{1}{2}+\varepsilon}(0,T;L^2(\Omega))}, \tag{2.13}$$

that, together with (2.12), says that Z_n^N is bounded, uniformly in n and N, in the spaces $H^{\frac{1}{2}-\varepsilon}(0,T;L^2(\Omega))$, $L^\infty(0,T;L^2(\Omega))$ and $L^2(0,T;H^1(\Omega))$.

Hence, it is possible to extract a (diagonal) subsequence converging weakly in $L^2(0,T;H^1(\Omega))$, weakly-$*$ in $L^\infty(0,T;L^2(\Omega))$ and strongly in $L^2((0,T) \times \Omega)$ to the unique solution Z of problem (2.9). Indeed, $Z \in H^{\frac{1}{2}-\varepsilon}(0,T;L^2(\Omega))$, that is, the topological dual space of $H^{-\frac{1}{2}+\varepsilon}(0,T;L^2(\Omega))$, the space to which $\partial_t G$ belongs to. Furthermore, by passing to the limit and by using the semi-continuity of the norms, it follows that Z satisfies the claim stated at the beginning of this proof. $\square$

REMARK 2.3. *The assumption $g \in H^{\frac{1}{2}+\varepsilon}(0,T;H^{\frac{1}{2}}(\Gamma))$, which in turn gives $G \in H^{\frac{1}{2}+\varepsilon}(0,T;H^1(\Omega))$, seems to be rather technical, for the presence of ϵ. If $g \in H^{\frac{1}{2}}(0,+\infty;H^{\frac{1}{2}}(\Gamma))$, it follows that $G \in H^{\frac{1}{2}}(0,+\infty;H^1(\Omega))$ and Theorem 2.2 holds accordingly.*

Indeed, the main point is estimate (2.12), in which the right-hand side becomes $\|G^N\|_{H^{\frac{1}{2}}}\|Z_n^N\|_{H^{\frac{1}{2}}}$ and, following the lines of the proof presented above, the estimate on the Fourier transform gives that Z_n^N is bounded, uniformly in n and N, in $H^{\frac{1}{2}}(0,+\infty;L^2(\Omega))$. Notice that in the critical case $H^{\frac{1}{2}}$, we work on the whole time

interval $[0, +\infty)$, *to avoid the boundary terms* $\|G^N(0)\|$ *and* $\|G^N(T)\|$, *that cannot be estimated by using the Morrey inequality.*

We also note that this small relaxation on the assumptions on G *requires to add the hypothesis* $G \in L^\infty(0, T; L^2(\Omega))$. *Otherwise the function* z *will not belong itself to* $L^\infty(0, T; L^2(\Omega))$ *and this fact is crucial to prove the corresponding bound for weak solutions to the full nonlinear Navier-Stokes problem.*

2.4. The non-linear problem. In this section we finally prove Theorem 1.1. Again, we make use of an auxiliary problem; namely, we introduce the new variables

$$U = u - z \qquad \text{and} \qquad P = p - q,$$

where (z, q) is the solution to the linear evolution problem (2.8), and the pair (U, P) solves the following problem

$$\begin{cases} \partial_t U - \nabla \cdot \mathbb{D}(U) + [(U + z) \cdot \nabla](U + z) + \nabla P = f & \text{in } \Omega \times (0, T) \\ \nabla \cdot U = 0 & \text{in } \Omega \times (0, T) \\ U \cdot \underline{\mathrm{n}} = 0 & \text{on } \Gamma \times (0, T) \\ \delta \underline{\mathrm{n}} \cdot \mathbb{D}(U) \cdot \underline{\tau} + U \cdot \underline{\tau} = 0 & \text{on } \Gamma \times (0, T) \\ U(x, 0) = u_0(x) - G(x, 0) & \text{in } \Omega. \end{cases} \qquad (2.14)$$

By virtue of Theorem 2.2, the existence Theorem 1.1 for the non-linear problem is a straightforward consequence of the following proposition.

PROPOSITION 2.4. *Assume that* (G, Π) *is solution to system* (2.4), *with* $G \in H^{\frac{1}{2}+\varepsilon}(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$, *then there is a unique*

$$U \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \qquad (2.15)$$

solution to problem (2.14). *Moreover, the following estimate holds true:*

$$\sup_{0 \le s \le t} \|U(s)\|^2 + \int_0^t \left(\|\mathbb{D}(U)\|^2 + \frac{1}{\delta}\|U\|_\Gamma^2 \right) ds \le$$

$$\le \|u_0 - G(\cdot, 0)\|^2 \mathrm{e}^{A(t)} + C \int_0^t (\|f\|^2 + \|\nabla z(s)\|^2 \|z(s)\|) \mathrm{e}^{A(t) - A(s)} ds, \quad (2.16)$$

where

$$A(t) = Ct + C \left(1 + \|z\|_{L^\infty(0, T; L^2(\Omega))}^2 \right) \int_0^t \|\nabla z\|^2 ds,$$

and C *is a constant depending only on* Ω.

Proof. The proof is rather standard and proceeds *via* a Faedo-Galerkin approximation, as in the proof of Theorem 2.2. We only show an *a-priori* estimate, whose computations are formal, but are completely meaningful at the level of the Faedo-Galerkin approximate functions. Multiply (2.14) by U and integrate by parts, to get

$$\frac{1}{2}\frac{d}{dt}\|U\|^2 + \|\mathbb{D}(U)\|^2 + \frac{1}{\delta}\|U\|_\Gamma^2 = \int_\Omega U \cdot [(U + z) \cdot \nabla](U + z) + \int_\Omega f \cdot U.$$

The estimate of the integral involving f is straightforward, since it is bounded by $\|f\|^2 + \|U\|^2$. We estimate the non-linear term in the right-hand side by using the

Gagliardo-Nirenberg inequality[4],

$$\|u\|_{L^4} \le C\|u\|^{1/2}\|\nabla u\|^{1/2} \qquad \forall\, u \in H^1(\Omega). \tag{2.17}$$

We first observe that, since $\nabla \cdot U = 0$ and $U \cdot \underline{\mathrm{n}} = 0$,

$$\int_\Omega U \cdot (U \cdot \nabla)U = 0 \qquad \text{and} \qquad \int_\Omega U \cdot (U \cdot \nabla)z = -\int_\Omega z \cdot (U \cdot \nabla)U,$$

so that, using repeatedly the Gagliardo-Nirenberg inequality above given, Hölder's inequality and Young's inequality, we get

$$
\begin{aligned}
\left| \int_\Omega U \cdot [(U+z) \cdot \nabla](U+z) \right| &\le 2\|U\|_{L^4}\|z\|_{L^4}\|\nabla U\| + \|U\|_{L^4}\|z\|_{L^4}\|\nabla z\| \\
&\le \frac{1}{2}\|\nabla U\|^2 + C\|z\|_{L^4}^4\|U\|^2 + C\|z\|_{L^4}^{\frac{4}{3}}\|\nabla z\|^{\frac{4}{3}}\|U\|^{\frac{2}{3}} \\
&\le \frac{1}{2}\|\nabla U\|^2 + C\|z\|^2\|\nabla z\|^2\|U\|^2 + C\|z\|^{\frac{2}{3}}\|\nabla z\|^2\|U\|^{\frac{2}{3}} \\
&\le \frac{1}{2}\|\nabla U\|^2 + C(1+\|z\|^2)\|\nabla z\|^2\|U\|^2 + C\|\nabla z\|^2\|z\|.
\end{aligned}
$$

Since, by Theorem 2.2, $z \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$, both terms $(1 + \|z\|^2)\|\nabla z\|^2$ and $\|\nabla z\|^2\|z\|$ are integrable in time and, by Gronwall's lemma, we can deduce that U is bounded in $L^\infty(0,T;L^2(\Omega))$ and $L^2(0,T;H^1(\Omega))$. Moreover, formula (2.16) also follows.

Finally, uniqueness of the solution follows from similar arguments. Indeed, if $\widetilde{U}$ is the difference between two solutions U_1 and U_2, one easily gets

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{U}\|^2 \le (\|U_2\|_{L^4}^4 + \|z\|_{L^4}^4 + \|\nabla z\|^2)\|\widetilde{U}\|^2$$

and, since $\widetilde{U}(0) = 0$, from Gronwall's lemma it follows that $\widetilde{U} \equiv 0$. $\square$

REMARK 2.5. *In the proof of the result of this section we used in a fundamental way estimate (2.17). In the three-dimensional case this inequality is not anymore true. Instead, it holds*

$$\|u\|_{L^4} \le C\|u\|^{1/4}\|\nabla u\|^{3/4} \qquad \forall\, u \in H^1(\Omega),$$

and the latter can be used to prove just local existence of weak solutions. The global result proved in the 2D case stems in an essential manner on the stronger estimate and this is the critical difference between the two cases.

3. The proof of Theorem 1.2. In this section we study the limit of the solution of the vorticity seeding model as $\delta \to 0$. We prove the convergence of the solutions of (3.1) to the solutions of the non-stationary Navier-Stokes system with the no-slip boundary condition (3.2). This supports the idea of using a non-standard boundary condition on the new boundary $\Gamma_1 \subset \Omega$, such that the region between Γ and Γ_1 is very narrow. When the width of this region (presumably the boundary layer) shrinks to zero, then the solution of the Navier-Stokes equations with the usual no-slip boundary condition is recovered.

---

[4]Note that such a inequality is a little bit more general than the so-called *Ladyžhenskaya inequality*, since the functions are not vanishing on the boundary of Ω.

3.1. Comparison of solutions with different boundary data. Let $g \in H^{\frac{1}{2}+\varepsilon}(0,T; H^{\frac{1}{2}}(\Gamma))$, satisfying the compatibility condition (1.5), $f \in L^2((0,T) \times \Omega)$ and $u_0 \in H^1(\Omega)$, with $\nabla \cdot u_0 = 0$. Without loss of generality, we can assume that $u_0 \equiv 0$. Denote by G_δ the solution of the linear stationary problem (2.4), with boundary condition $G_\delta \cdot \underline{n} = \delta\, g$. Consider the solution (u_δ, p_δ) to the system

$$\begin{cases} \partial_t u_\delta - \nabla \cdot \mathbb{D}(u_\delta) + u_\delta \cdot \nabla u_\delta + \nabla p_\delta = f & \text{in } \Omega \times (0,T) \\ \nabla \cdot u_\delta = 0 & \text{in } \Omega \times (0,T) \\ u_\delta \cdot \underline{n} = \delta\, g(x,t) & \text{on } \Gamma \times (0,T) \\ \delta\, \underline{n} \cdot \mathbb{D}(u_\delta) \cdot \underline{\tau} + u_\delta \cdot \underline{\tau} = 0 & \text{on } \Gamma \times (0,T) \\ u_\delta(x,0) = u_0(x) + G(x,0) & \text{in } \Omega, \end{cases} \tag{3.1}$$

We want to show the convergence of the vector valued function u_δ to the solution v of the Navier-Stokes equations with zero Dirichlet data:

$$\begin{cases} \partial_t v - \nabla \cdot \mathbb{D}(v) + v \cdot \nabla v + \nabla \pi = f & \text{in } \Omega \times (0,T) \\ \nabla \cdot v = 0 & \text{in } \Omega \times (0,T) \\ v = 0 & \text{on } \Gamma \times (0,T) \\ v(x,0) = u_0 & \text{in } \Omega. \end{cases} \tag{3.2}$$

To this end, we also introduce the solution z_δ to the linear evolution problem (2.8), with boundary condition $z_\delta \cdot \underline{n} = \delta\, g$, and we set

$$U_\delta = u_\delta - z_\delta \qquad \text{and} \qquad w_\delta = u_\delta - z_\delta - v.$$

The function w_δ satisfies the following homogeneous system

$$\begin{cases} \partial_t w_\delta - \nabla \cdot \mathbb{D}(w_\delta) + R(w_\delta, z_\delta, v, U_\delta) + \nabla r = 0 & \text{in } \Omega \times (0,T) \\ \nabla \cdot w_\delta = 0 & \text{in } \Omega \times (0,T) \\ w_\delta \cdot \underline{n} = 0 & \text{on } \Gamma \times (0,T) \\ \delta\, \underline{n} \cdot \mathbb{D}(w_\delta) \cdot \underline{\tau} + w_\delta \cdot \underline{\tau} = -\delta\, \underline{n} \cdot \mathbb{D}(v) \cdot \underline{\tau} & \text{on } \Gamma \times (0,T) \\ w_\delta(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{3.3}$$

where

$$R(w_\delta, z_\delta, v, U_\delta) = (U_\delta \cdot \nabla)w_\delta + (w_\delta \cdot \nabla)v + (U_\delta \cdot \nabla)z_\delta + (z_\delta \cdot \nabla)U_\delta + (z_\delta \cdot \nabla)z_\delta.$$

We multiply the first equation in (3.3) by w_δ and integrate by parts to get

$$\frac{1}{2}\frac{d}{dt}\|w_\delta\|^2 + \|\mathbb{D}(w_\delta)\|^2 + \frac{1}{\delta}\|w_\delta\|_\Gamma^2 = -\int_\Gamma (\underline{n} \cdot \mathbb{D}(v) \cdot \underline{\tau})(w_\delta \cdot \underline{\tau})d\sigma - \int_\Omega w_\delta \cdot R(w_\delta, z_\delta, v, U_\delta).$$

The boundary integral in the right-hand side may be increased in the following way,

$$\left| \int_\Gamma (\underline{n} \cdot \mathbb{D}(v) \cdot \underline{\tau})(w_\delta \cdot \underline{\tau})\, d\sigma \right| \leq \frac{\delta}{2}\|v\|_{H^2}^2 + \frac{1}{2\delta}\|w_\delta\|_\Gamma^2. \tag{3.4}$$

We analyse then the second integral. By integration by parts over Ω (note that $U_\delta \cdot \underline{n} = 0$ on Γ) and by using Hölder's inequality, the Gagliardo-Nirenberg inequality,

and Young's inequality, we obtain

$$\left| \int_\Omega w_\delta \cdot R(w_\delta, z_\delta, v, U_\delta)\, dx \right| \leq$$

$$\leq \frac{1}{2}\|\mathbb{D}(w_\delta)\|^2 + C\big[\|\nabla v\|^2 + \|\nabla U_\delta\|^{\frac{2}{3}}\|\nabla z_\delta\|^{\frac{4}{3}} + \|\nabla U_\delta\|^{\frac{4}{3}}\|\nabla z_\delta\|^{\frac{2}{3}} + \|\nabla z_\delta\|^2\big]\|w_\delta\|^2$$

$$+ C\big[\|\nabla U_\delta\|^{\frac{2}{3}}\|\nabla z_\delta\|^{\frac{4}{3}}\|U_\delta\| + \|\nabla U_\delta\|^{\frac{4}{3}}\|\nabla z_\delta\|^{\frac{2}{3}}\|z_\delta\| + \|\nabla z_\delta\|^2\|z_\delta\|\big].$$

Indeed, for example,

$$\left| \int_\Omega w_\delta \cdot (z_\delta \cdot \nabla)U_\delta \right| \leq \|w_\delta\|_{L^4}\|z_\delta\|_{L^4}\|\nabla U\|$$

$$\leq C\|w_\delta\|^{\frac{1}{2}}\|\mathbb{D}(w_\delta)\|^{\frac{1}{2}}\|z_\delta\|^{\frac{1}{2}}\|\mathbb{D}(z_\delta)\|^{\frac{1}{2}}\|\nabla U_\delta\|$$

$$\leq \frac{1}{8}\|\mathbb{D}(w_\delta)\|^2 + C\|\nabla U_\delta\|^{\frac{4}{3}}\|\nabla z_\delta\|^{\frac{2}{3}}\|w_\delta\|^{\frac{2}{3}}\|z_\delta\|^{\frac{2}{3}}$$

$$\leq \frac{1}{8}\|\mathbb{D}(w_\delta)\|^2 + C\|\nabla U_\delta\|^{\frac{4}{3}}\|\nabla z_\delta\|^{\frac{2}{3}}(\|w_\delta\|^2 + \|z_\delta\|).$$

For simplicity we set

$$\psi(t) := C\big[\|\nabla U_\delta\|^{\frac{2}{3}}\|\nabla z_\delta\|^{\frac{4}{3}}\|U_\delta\| + \|\nabla U_\delta\|^{\frac{4}{3}}\|\nabla z_\delta\|^{\frac{2}{3}}\|z_\delta\| + \|\nabla z_\delta\|^2\|z_\delta\|\big],$$

$$\phi(t) := \big[\|\nabla v\|^2 + \|\nabla U_\delta\|^{\frac{2}{3}}\|\nabla z_\delta\|^{\frac{4}{3}} + \|\nabla U_\delta\|^{\frac{4}{3}}\|\nabla z_\delta\|^{\frac{2}{3}} + \|\nabla z_\delta\|^2\big],$$

$$\chi(t) := \|v\|_{H^2}^2,$$

so that, by recollecting all together the estimates we have before obtained, we end up with the following differential inequality

$$\frac{d}{dt}\|w_\delta\|^2 + \|\mathbb{D}(w_\delta)\|^2 + \frac{1}{\delta}\|w_\delta\|_\Gamma^2 \leq \delta\chi(t) + \psi(t) + \phi(t)\|w_\delta\|^2 \tag{3.5}$$

3.1.1. Estimate of ψ and ϕ in terms of δ. First, we note that the functions ψ and ϕ belong to $L^1(0, T)$, by virtue of the regularity properties (2.10) and (2.15) of z_δ and U_δ, respectively. The function $\chi(t)$ belongs to $L^1(0, T)$ as well, since

$$\|v\|_{L^2(0,T;H^2)} \leq C[\|v(x,0)\|_{H^1} + \|f\|_{L^2((0,T)\times\Omega)}],$$

see for instance Constantin and Foiaş [9]. And also in this point is crucial to consider the 2D problem, since such estimate is available just for small times in the 3D case.

It is important now to sharply check the correct behaviour of the functions ϕ and $\psi(t)$ in terms of δ. Before going further, we collect in the following lemma the results of Section 2 that we shall need.

LEMMA 3.1. *Under the assumptions of Theorem 1.2, the following estimates hold*

$$(i) \quad \sup_{0\leq t\leq T}\|z_\delta\|^2 + \int_0^T \|\mathbb{D}(z_\delta)\|^2\, dt \leq C_1\delta\|g\|_{H^{\frac{1}{2}+\varepsilon}(0,T;H^{\frac{1}{2}}(\Gamma))}^2,$$

$$(ii) \quad \sup_{0\leq t\leq T}\|U_\delta\|^2 + \int_0^T \left(\|\mathbb{D}(U_\delta)\|^2 + \frac{1}{\delta}\|U_\delta\|_\Gamma^2\right) dt \leq C_2,$$

for all $0 < \delta \leq 1$, where the constant C_1 depends only on Ω and the constant C_2 depends only on Ω, T, $\|f\|_{L^2}$, and $\|g\|_{H^{\frac{1}{2}+\varepsilon}}$.

Proof. Property *(i)* is a consequence of (2.11). Indeed, by (2.6) and Poincaré inequality,

$$\|G_\delta\| \leq C\|\mathbb{D}(G_\delta)\| \leq C\sqrt{\delta}\|g\|,$$

so that $\|G_\delta\|_{H^{\frac{1}{2}+\varepsilon}(0,T;L^2(\Omega))} \leq C\sqrt{\delta}\|g\|_{H^{\frac{1}{2}+\varepsilon}(0,T;L^2(\Omega))}$.

Again, from (2.6),

$$\int_0^T \|\mathbb{D}(G_\delta)\|^2 \, dt \leq C\delta\|g\|^2_{H^{\frac{1}{2}+\varepsilon}(0,T;L^2(\Omega))},$$

and in conclusion *(i)* holds true.

As it concerns *(ii)*, we have that, by Poincaré inequality, (2.6) and Sobolev embeddings,

$$\|G_\delta(\cdot,0)\| \leq C\|\mathbb{D}(G_\delta)(\cdot,0)\| \leq C\sqrt{\delta}\|g(\cdot,0)\| \leq C\sqrt{\delta}\|g\|_{H^{\frac{1}{2}+\varepsilon}(0,T;L^2)}.$$

Moreover, since $\delta \leq 1$ and from the above estimates it follows

$$A(t) \leq A(T) \leq CT + C(1 + \|g\|^4_{H^{\frac{1}{2}+\varepsilon}(0,T;L^2(\Gamma))})$$

that is, $A(t)$ is uniformly bounded by a constant independent of δ, and

$$\int_0^t (\|f\|^2 + \|\nabla z_\delta\|^2 \|z_\delta\|) \mathrm{e}^{A(t)-A(s)} \, ds \leq \mathrm{e}^{A(T)} \Big(\int_0^T \|f\|^2 + \|z_\delta\|_{L^\infty(0,T;L^2)} \|\mathbb{D}(z_\delta)\|^2 \Big)$$

$$\leq (\|f\|^2_{L^2((0,T)\times\Omega)} + C\|g\|^3_{H^{\frac{1}{2}+\varepsilon}}) \mathrm{e}^{A(T)}$$

as well, so that, by (2.16), also *(ii)* follows. □

The first consequence of the above lemma is that

$$\int_0^t \phi(s) \, ds \leq \int_0^T \|\nabla v\|^2 \, dt + C(\Omega, T, f, g),$$

with a bound uniform in δ, for δ small. Moreover

$$\int_0^t \psi(s) \, ds \leq C \left(\int_0^T \|\nabla U_\delta\|^2 \right)^{\frac{1}{3}} \left(\int_0^T \|\nabla z_\delta\|^2 \right)^{\frac{2}{3}} \|U_\delta\|_{L^\infty}$$

$$+ C \left(\int_0^T \|\nabla U_\delta\|^2 \right)^{\frac{2}{3}} \left(\int_0^T \|\nabla z_\delta\|^2 \right)^{\frac{1}{3}} \|z_\delta\|_{L^\infty} + C\|z_\delta\|_{L^\infty} \int_0^T \|\nabla z_\delta\|^2$$

$$\leq C(\delta^{\frac{2}{3}} + \delta^{\frac{5}{6}} + \delta^{\frac{3}{2}})$$

$$\leq C\delta^{\frac{2}{3}},$$

so that, by (3.5) and Gronwall's inequality, it follows that

$$\sup_{0 \leq t \leq T} \|w_\delta\|^2 + \int_0^T \left(\|\mathbb{D}(w_\delta)\|^2 + \frac{1}{\delta}\|w_\delta\|^2_\Gamma \right) dt \leq \int_0^T [\delta\chi(s) + \psi(s)] \mathrm{e}^{\int_s^T \phi(r) \, dr} \, ds$$

$$\leq \mathrm{e}^{\int_0^T \phi(s) \, ds} \int_0^T [\delta\chi(s) + \psi(s))] \, ds$$

$$\leq C\delta^{\frac{2}{3}}$$

and finally

$$\sup_{0 \le t \le T} \|w_\delta\|^2 + \int_0^T \left(\|\mathbb{D}(w_\delta)\|^2 + \frac{1}{\delta}\|w_\delta\|_\Gamma^2 \right) dt = \mathcal{O}(\delta^{\frac{2}{3}}).$$

Since the same norms of z_δ are of order $\delta^{\frac{1}{2}}$, by the previous lemma, we finally deduce that $u_\delta - v$ is of the order $\delta^{\frac{1}{3}}$, and the theorem is proved.

REMARK 3.2. The tangential trace of the function w_δ converges a little better, since from the above estimate we can deduce

$$\|w_\delta\|_{L^2(\Gamma \times (0,T))} = \mathcal{O}(\delta^{5/6}).$$

REMARK 3.3. The estimates show that there is a crucial loss in the estimates due 1) to the boundary effect, in the estimate (2.6); 2) to the non-linearity (recall the contribution of U_δ in the estimate of $\psi(t)$). We observe that the original model was supplemented by the following boundary condition

$$u \cdot \underline{n} = \delta^2 g,$$

but we used in fact $u \cdot \underline{n} = \delta\, g$. By running through the proof of Theorem 1.2 one gets, using the boundary condition

$$u \cdot \underline{n} = \delta^\alpha g,$$

that $z_\delta = \mathcal{O}(\delta^{\alpha - \frac{1}{2}})$, so that $\int \psi(t)\, dt = \mathcal{O}(\delta^{\frac{4}{3}(\delta - \frac{1}{2})})$ and finally $w_\delta = \mathcal{O}(\delta^{\frac{2}{3}(\alpha - \frac{1}{2})})$. In particular, for $\alpha = 2$, that is, the value corresponding to the original model, the behaviour of z_δ matches the loss in the estimates due to the boundary term (3.4). Hence, for larger values of α, the convergence remains slow, because of this term. Supposedly, the loss in the convergence rate caused by this term is an intrinsic feature of the problem, while those caused by the non-linearity may have technical reasons.

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